SHARPER UPPER BOUNDS FOR THE ORDER OF APPEARANCE IN THE FIBONACCI SEQUENCE

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Abstract. Let \( F_n \) be the \( n \)th Fibonacci number. The order of appearance \( z(n) \) of a natural number \( n \) is defined as the smallest natural number \( k \) such that \( n \) divides \( F_k \). In 1975, J. Sallé proved that \( z(n) \leq 2n \), for all positive integers \( n \). In this paper, we shall provide sharper upper bounds for \( z(n) \) which are substantially smaller than \( 2n \) for some values of \( n \). Moreover, we shall prove that

\[
\liminf_{n \to \infty} \frac{z(n)}{n} = 0.
\]

1. Introduction

Let \((F_n)_{n \geq 0}\) be the Fibonacci sequence given by

\[
F_{n+2} = F_{n+1} + F_n, \quad \text{for} \quad n \geq 0, \quad \text{where} \quad F_0 = 0 \quad \text{and} \quad F_1 = 1.
\]

These numbers are well-known for possessing amazing properties (consult [6] together with its very extensive annotated bibliography for additional references and history).

The study of the divisibility properties of Fibonacci numbers has always been a popular area of research. For example, it is still an open problem to decide if there are infinitely many primes in that sequence. Let \( n \) be a positive integer, the order (or rank) of appearance of \( n \) in the Fibonacci sequence, denoted by \( z(n) \), is defined as the smallest positive integer \( k \), such that \( n \mid F_k \) (some authors also call it order of apparition, or Fibonacci entry point). This function can be implemented in Mathematica [22] as

\[
z[n_] := \text{Catch}[\text{Do}[i; \text{If}[\text{Mod}[\text{Fibonacci}[i], n] == 0, \text{Throw}[i], \{i, 2n\}]]]
\]

There are several results about \( z(n) \) in the literature. For instance, in 1878, E. Lucas showed that \( z(n) < \infty \) for all \( n \geq 1 \). The proof of this fact is an immediate consequence of the Théorème Fondamental of Section XXVI in [8, p. 300]. We remark that there is no known closed formula for \( z(n) \) and so Diophantine equations related to \( z(n) \) are one of the best tools for understanding its behavior. This function gained great attention in 1992, when Z. H. Sun and Z. W. Sun [20] proved that to show that all solutions of the Diophantine equation \( z(n) = z(n^2) \) are composite numbers, would imply the Fermat’s Last Theorem. It is known that there are no prime solutions when \( n < 3.23 \cdot 10^{15} \) (PrimeGrid Project, May 2012).

Recently, the author wrote a series of papers related to \( z(n) \). We refer the reader to [9, 10, 11, 12, 14] for explicit formulas for the order of appearance of integers related to Fibonacci and Lucas numbers, such as \( F_m \pm 1, F_n F_{n+1} F_{n+2}, F_n^k \) and \( L_n L_{n+1} L_{n+2} \). Also, solutions for the Diophantine equation \( z(n) = n + \ell \), with \( \ell \geq 0 \), were studied in [13, 15, 16]. For instance, for \( \ell = 0 \), the solutions are of the form \( 5^k \) or \( 12 \cdot 5^k \) (\( k \geq 0 \)), for \( \ell = 1 \), the solutions are prime numbers and for \( \ell = 2 \), the only solution is \( n = 4 \).

Concerning upper bounds for \( z(n) \), one can apply Dirichlet’s Box Principle to get the bound \( z(n) \leq (n - 1)^2 + 1 \) (see [21, Theorem, p. 52]). In the case of a prime number \( p \), one has the better bound \( z(p) \leq p + 1 \).

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In 1975, J. Sallé [18] proved that \( z(n) \leq 2n \), for all positive integers \( n \). This is the sharpest upper bound for \( z(n) \), since for example, \( z(6) = 12 \) and \( z(30) = 60 \). Actually, proceeding along the same lines as the proof of Theorem 1.1 of [13], one obtains that

\[
z(n) = 2n \quad \text{if and only if} \quad n = 6 \cdot 5^k, \quad \text{for} \quad k \geq 0.
\]  

(1.1)

However, apart from these cases this upper bound is very weak. For instance, \( z(3731) = 280 < 0.08 \cdot 3731 \).

We still remark that Sallé’s proof depends, strongly, on a result due to Carmichael [2, Theorem XIII].

In this paper, we shall combine a refinement of the method in [18] together with the author’s approach [13] (using formulae for the \( p \)-adic order of Fibonacci numbers) to get substantially better upper bounds for \( z(n) \), when \( n \neq 6 \cdot 5^k \) is a composite number. This proof does not depend on the Carmichael result. Moreover, the improvement depends on the number of distinct prime factors of \( n \), denoted by \( \omega(n) \).

Our main results are the following:

**Theorem 1.1.** We have

(i) \( z(2^k) = 3 \cdot 2^{k-2} \) (for \( k \geq 3 \)), \( z(3^k) = 4 \cdot 3^{k-1} \) (for \( k \geq 1 \)) and \( z(5^k) = 5^k \) (for \( k \geq 0 \)).

(ii) If \( p > 5 \) is a prime, then

\[
z(p^k) \leq \left( p - \frac{5}{p} \right) p^{k-1}, \quad \text{for} \quad k \geq 1,
\]

where, as usual, \( \left( \frac{a}{q} \right) \) denotes the Legendre symbol of \( a \) with respect to a prime \( q > 2 \).

For the cases when \( \omega(n) \geq 2 \), we proved that

**Theorem 1.2.** Let \( n \) be an odd integer with \( \omega(n) \geq 2 \), then

\[
z(n) \leq 2 \cdot \left( \frac{2}{3} \right)^{\omega(n) - \delta_n} n,
\]

where

\[
\delta_n = \begin{cases} 
0, & \text{if } 5 \nmid n; \\
1, & \text{if } 5 \mid n.
\end{cases}
\]

**Theorem 1.3.** Let \( n \) be an even integer with \( \omega(n) \geq 2 \), we have that

(i) If \( \nu_2(n) \geq 4 \), then

\[
z(n) \leq \frac{3}{4} \cdot \left( \frac{2}{3} \right)^{\omega(n) - \delta_{n-1}} n.
\]

(ii) If \( \nu_2(n) = 1 \), then

\[
z(n) \leq \begin{cases} 
3n/2, & \text{if } \omega(n) = 2 \text{ and } 5 \mid n; \\
2n, & \text{if } \omega(n) = 2 \text{ and } 5 \nmid n; \\
3 \cdot (2/3)^{\omega(n) - \delta_{n-1}} n, & \text{if } \omega(n) > 2.
\end{cases}
\]

(iii) If \( \nu_2(n) \in \{2,3\} \), then

\[
z(n) \leq \begin{cases} 
3n/2, & \text{if } \omega(n) = 2 \text{ and } 5 \mid n; \\
\left( \frac{2}{3} \right)^{\omega(n) - \delta_2 - 2} n, & \text{if } \omega(n) > 2.
\end{cases}
\]

where \( \nu_2(n) \) is the 2-adic valuation.
We organize the paper as follows. In Section 2, we will recall some useful properties of Fibonacci numbers such as a result concerning the $p$-adic order of $F_n$. Section 3 will be devoted to the proof of theorems. In the last section, we shall discuss the behavior of $z(n)/n$ when $n$ increases.

2. Auxiliary results

Before proceeding further, we state some facts on Fibonacci numbers for the convenience of the reader.

Lemma 2.1. We have

(a) $n \mid m$ if and only if $F_n \mid F_m$.
(b) $F_{p - \frac{\nu_p}{p}} \equiv 0 \pmod{p}$, for all primes $p$.

Proofs of these assertions can be found in [6]. We refer the reader to [1, 5, 17] for more details and additional bibliography.

The second lemma is a consequence of the previous one.

Lemma 2.2. (Cf. Lemma 2.2 (c) of [10]) If $n \mid F_m$, then $z(n) \mid m$.

Note that Lemma 2.1 (b) and Lemma 2.2 implies that $z(p)$ divides $p - (5/p)$, for all primes $p$. In particular, $z(p) \leq p + 1$ for all primes $p$.

The $p$-adic order (or valuation) of $r$, $\nu_p(r)$, is the exponent of the highest power of a prime $p$ which divides $r$. Throughout the paper, we shall use the known facts that $\nu_p(ab^s) = \nu_p(a) + \epsilon \nu_p(b)$, for $\epsilon \in \{-1, 1\}$, and that $a \mid b$ if and only if $\nu_p(a) \leq \nu_p(b)$, for all primes $p$.

We remark that the $p$-adic order of Fibonacci numbers was completely characterized, see [4, 7]. For instance, from the main results of Lengyel [7], we extract the following result.

Lemma 2.3. For $n \geq 1$, we have

$$\nu_2(F_n) = \begin{cases} 
0, & \text{if } n \equiv 1, 2 \pmod{3}; \\
1, & \text{if } n \equiv 3 \pmod{6}; \\
3, & \text{if } n \equiv 6 \pmod{12}; \\
\nu_2(n) + 2, & \text{if } n \equiv 0 \pmod{12},
\end{cases}$$

$$\nu_5(F_n) = \nu_5(n), \text{ and if } p \text{ is prime } \neq 2 \text{ or } 5, \text{ then }$$

$$\nu_p(F_n) = \begin{cases} 
\nu_p(n) + \nu_p(F_{z(p)}), & \text{if } n \equiv 0 \pmod{z(p)}; \\
0, & \text{if } n \not\equiv 0 \pmod{z(p)}.
\end{cases}$$

A proof for this result can be found in [7].

As usual, from now on we use the well-known notation $[a, b] = \{a, a + 1, \ldots, b\}$, for integers $a < b$.

Now we are ready to deal with the proofs of our results.

3. The proofs

3.0.1. Proof of Theorem 1.1. (i) By Theorem 1.1 of [12], we have that $z(F_n^k) = nF_{n-1}^k/2$, for $k \geq 3$ and $n \equiv 3 \pmod{6}$, and also $z(F_n^{k+1}) = nF_n^k$, for $k \geq 0$ and $n \equiv 3 \pmod{6}$. Since $2 = F_3$, $3 = F_4$ and $5 = F_5$, one can easily use the previous formulas to get the desired result.

(ii) By Lemma 2.2, it suffices to prove that $p^k \mid F_{(p-(5/p))p^{k-1}}$. This follows from the fact that

$$\nu_p(F_{(p-(5/p))p^{k-1}}) = \nu_p((p-(5/p))p^{k-1}) + \nu_p(F_{z(p)})$$

$$= k - 1 + \nu_p(F_{z(p)}) \geq k = \nu_p(p^k).$$
where we used Lemma 2.3 together with \( \nu_p(F_z(p)) \geq 1. \)

3.0.2. Proof of Theorem 1.2. Write \( n = 5^a p_1^{a_1} \cdots p_k^{a_k} \), where \( p_i \in \{2, 5\} \) is prime (\( p_i \neq p_j \), if \( i \neq j \)) and \( a_i \geq 1 \), for all \( i \in [1, k] \). Setting \( z_i = p_i - (5/p_i) \), we have that \( z_i \mid z_i \), for \( i \in [1, k] \). We claim that \( n \mid F_m \), where

\[
m := 2 \cdot 5^a \left( \frac{z_1 p_1^{a_1-1}}{2} \right) \cdots \left( \frac{z_k p_k^{a_k-1}}{2} \right).
\]

Note that \( m \) is well defined, because \( z_i \) is even for all \( i \in [1, k] \). Since \( 5^a, p_1^{a_1}, \ldots, p_k^{a_k} \) are pairwise coprime and \( 5^a \mid F_m \) (keep in mind that \( \nu_5(F_m) = \nu_5(m) = a \)), it suffices to prove that \( p_i \mid F_m \), or equivalently, \( \nu_{p_i}(F_m) \geq a_i \), for all \( i \in [1, k] \). First, observe that \( z(p_i) \mid m \), because \( m \) can be written as

\[
5^a \left( \frac{z_1 p_1^{a_1-1}}{2} \right) \cdots \left( z_k p_k^{a_k-1} \right).
\]

Therefore, Lemma 2.3 gives

\[
\nu_{p_i}(F_m) = \nu_{p_i}(m) + \nu_{p_i}(F_z(p_i)) = a_i - 1 + \sum_{j=1}^{k} \nu_{p_i}(z_j) + \nu_{p_i}(F_z(p_i)) \geq a_i.
\]

Thus \( n \mid F_m \) and so \( z(n) \leq m \) (by Lemma 2.2). However,

\[
\frac{m}{n} = 2 \cdot \left( \frac{z_1}{2p_1} \right) \cdots \left( \frac{z_k}{2p_k} \right) \leq 2 \cdot \left( \frac{p_1 + 1}{2p_1} \right) \cdots \left( \frac{p_k + 1}{2p_k} \right) \leq 2 \cdot \left( \frac{2}{3} \right)^k,
\]

where we used that \( z(p_i) \leq p_i + 1 \) and that \( (p_i + 1)/2p_i \leq 2/3 \) (since \( p_i \geq 3 \)). The previous inequality together with that fact that \( \omega(n) = k + \delta_n \) yields

\[
(z(n) \leq m \leq 2 \cdot \left( \frac{2}{3} \right)^{\omega(n)-\delta_n} n.
\]

3.0.3. Proof of Theorem 1.3. (i) If \( n = 2^a 5^b p_1^{a_1} \cdots p_k^{a_k} \), we choose

\[
m := 3 \cdot 2^{a-2} \cdot 5^b \left( \frac{z_1 p_1^{a_1-1}}{2} \right) \cdots \left( \frac{z_k p_k^{a_k-1}}{2} \right).
\]

Proceeding as in the proof of Theorem 1.2, one has that \( 5^b, p_1^{a_1}, \ldots, p_k^{a_k} \mid F_m \). In order to prove that \( n \mid F_m \), it is therefore enough to show that \( 2^a \) divides \( F_m \). Indeed, since \( a \geq 4 \), then \( 12 \mid m \) and Lemma 2.3 yields

\[
\nu_2(F_m) = \nu_2(m) + 2 = a + \sum_{j=1}^{k} (\nu_2(z_j) - 1) \geq a = \nu_2(2^a),
\]

where we used that \( \nu_2(z_i) \geq 1 \), for all \( i \in [1, k] \). As in the previous section, we get

\[
\frac{m}{n} \leq 3 \cdot \left( \frac{2}{3} \right)^k.
\]

The result follows because \( z(n) \leq m \) and \( \omega(n) = k + 1 + \delta_n. \)
(ii) and (iii) These items can be proved similarly, by suitable choices of \( m \) in each case. The only case requiring further analysis occurs when \( \nu_2(n) \in \{2, 3\} \) and \( \omega(n) > 2 \). For that, we have \( n = 2^a 5^b p_1^{m_1} \cdots p_k^{m_k} (a \in \{2, 3\}) \) while \( m \) can be chosen as

\[
3 \cdot 2^{a-1} \cdot 5^b \left( \frac{z_1 p_1^{a_1-1}}{2} \right) \cdots \left( \frac{z_k p_k^{a_k-1}}{2} \right).
\]

Since \( a - 1 \geq 1 \), then \( 6 \mid n \) and Lemma 2.3 gives

\[
\nu_2(F_m) \geq 3 \geq a = \nu_2(n).
\]

The upper bound for \( m/n \) is \( \left( \frac{3}{2} \right)^k \left( \frac{2}{3} \right)^{k-1} \) and the result follows since \( \omega(n) = k + 1 + \delta_n \).

The proof of Theorem 1.3 is then complete. \( \square \)

4. ON THE BEHAVIOR OF \( z(n)/n \)

In this section, we shall discuss the quotient \( z(n)/n \), for \( n \geq 1 \). A few approximated values of this sequence are

\[
1, 1.5, 1.333, 1.5, 1, 1.142, 0.75, 1.333, 1.5, 0.909, 1, 0.538, 1.714, 1.333, \ldots
\]

Clearly this sequence is not convergent (since \( z(2^k)/2^k = 3/4 \) and \( z(3^k)/3^k = 4/3 \), for all \( k \geq 3 \)). However, using the equivalence in (1.1) together with the fact that \( z(n)/n \leq 2 \), for all \( n \), we deduce that

\[
\limsup_{n \to \infty} \frac{z(n)}{n} = 2.
\]

But what is the value of \( \liminf_{n \to \infty} z(n)/n \)? Our final result provides an answer to this question.

**Proposition 4.1.** We have that

\[
\liminf_{n \to \infty} \frac{z(n)}{n} = 0.
\]

Before the proof, we recall that \( p_n\# \) denotes the \( n \)th primorial number which is defined as the product of the first \( n \) prime numbers. For instance, the values of \( p_n\# \) for \( n \in [1, 10] \) are

\[
2, 6, 30, 210, 2310, 30030, 510510, 9699690, 223092870, 6469693230, \ldots
\]

which is the OEIS [19] sequence A002110. To the best of our knowledge, the name primorial was coined in 1987 by Dubner [3].

**Proof.** Since \( z(n)/n \geq 0 \), then it suffices to prove that \( \lim_{n \to \infty} z(p_n\#)/p_n\# = 0 \). For that, note that Theorem 1.3 (ii) implies that \( z(p_n\#) \leq 3 \cdot (2/3)^{n-2} p_n\# \), for all \( n > 2 \) and therefore

\[
0 \leq \frac{z(p_n\#)}{p_n\#} \leq 3 \cdot \left( \frac{2}{3} \right)^{n-2}
\]

holds for all \( n > 2 \). Since \( \lim_{n \to \infty} (2/3)^{n-2} = 0 \), the Squeeze Theorem gives

\[
\lim_{n \to \infty} \frac{z(p_n\#)}{p_n\#} = 0
\]

and the proof is complete. \( \square \)

For example, \( z(n) < n/2013 \) if

\[
n = p_{24} = 23768741896345550770650537601358310.
\]
This shows that our bounds are effectively much better than $2n$, mainly when $\omega(n)$ is large.

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**References**


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