ON THE INTERSECTION OF TWO DISTINCT
\emph{k}-GENERALIZED FIBONACCI SEQUENCES

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Abstract. Let \( k \geq 2 \) and define \( F^{(k)} := (F_n^{(k)})_{n \geq 0} \), the \( k \)-generalized Fibonacci sequence whose terms satisfy the recurrence relation \( F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \ldots + F_{n-k}^{(k)} \), with initial conditions 0, 0, \ldots , 0, 1 ( \( k \) terms) and such that the first nonzero term is \( F_1^{(k)} = 1 \). The sequences \( F := F^{(2)} \) and \( T := F^{(3)} \) are the known Fibonacci and Tribonacci sequences, respectively. In 2005, Noe and Post made a conjecture related to the possible solutions of the Diophantine equation \( F_n^{(k)} = F_m^{(l)} \). In this note, we use transcendental tools to provide a general method for finding the intersections \( F^{(k)} \cap F^{(m)} \) which gives evidence supporting the Noe-Post conjecture. In particular, we prove that \( F \cap T = \{0, 1, 2, 13\} \).

Keywords: \( k \)-generalized Fibonacci numbers, linear forms in logarithms, reduction method

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1. Introduction

Several problems in number theory are actually questions about the intersection of two known sequences (or sets). Before giving examples, let us recall some terminology: let \( F := (F_n)_{n \geq 0} \) be the \emph{Fibonacci sequence}, \( \mathcal{P} := \{p: p \text{ prime}\} \), \( \mathcal{P} := \{y^t: y, t \in \mathbb{Z}, t > 1\} \) (the perfect powers), \( \mathcal{F} := \{n!: n \in \mathbb{Z}, n \geq 0\} \), \( \mathcal{R} := \{a(10^n - 1)/9: 1 \leq a \leq 9, n \in \mathbb{Z}, n > 0\} \) (the repdigits or unidigital numbers). Below, we cite some results about the intersection of these sets:

\( \triangleright \) Erdős and Selfridge [8] proved that \( \mathcal{F} \cap \mathcal{P} = \{1\} \).
\( \triangleright \) In 2000, Luca [25] proved that \( F \cap \mathcal{R} = \{0, 1, 2, 3, 5, 8, 55\} \).
\( \triangleright \) Luca [26] also proved that \( F \cap \mathcal{F} = \{1, 2\} \).
\( \triangleright \) In 2003, Bugeaud et al [4] showed that \( F \cap \mathcal{P} = \{0, 1, 8, 144\} \) (see [28] for a generalization).
Let \((a_n)_{n \geq 1}\) be the tower given by \(a_1 = 1\) and \(a_n = n^{a_{n-1}}\), for \(n \geq 2\). Luca and the author \([27]\) proved that \(\{a_1 + \ldots + a_n: n \geq 1\} \cap \mathcal{P} = \{1\}\).

However, some related questions are still open problems, as for instance the sets \(\mathbb{P} \cap F\) and \(\mathbb{P} \cap R\) are unknown.

Let \(k \geq 2\) and denote \(F^{(k)} := (F_n^{(k)})_{n \geq 0}\), the \(k\)-generalized Fibonacci sequence whose terms satisfy the recurrence relation

\[
F_n^{(k)} = F_{n-1}^{(k)} + F_{n-k}^{(k)} + \ldots + F_{n-k}^{(k)},
\]

with initial conditions \(0, 0, \ldots, 0, 1\) (\(k\) terms) and such that the first nonzero term is \(F_1^{(k)} = 1\).

The above sequence is one among the several generalizations of Fibonacci numbers. Such a sequence is also called \(k\)-step Fibonacci numbers, the Fibonacci \(k\)-sequence, or \(k\)-bonacci numbers. Clearly, for \(k = 2\) we obtain the well-known Fibonacci numbers and for \(k = 3\), Tribonacci numbers.

Recall that Tribonacci numbers have a long history. For the first time, they were studied in 1914 by Agronomoff \([1]\) and subsequently by many others. The name Tribonacci was coined in 1963 by Feinberg \([9]\). The basic properties of Tribonacci numbers can be found in \([18], [24], [36], [38]\). For recent papers, we refer the reader to \([3], [19], [20], [33]\) and to the collection \([21], [22], [23]\).

Recently, Alekseyev \([2]\) described how to compute the intersection of two Lucas sequences including the sequences of Fibonacci, Pell, Luca’s and Lucas-Pell numbers. In general, we refer the reader to \([34], [35], [37]\) for results on the intersection of two recurrence sequences.

In a very recent paper, Togbé and the author \([29]\) proved that only finitely many terms of a linear recurrence sequence whose characteristic polynomial has a dominant root can be repdigits. As an application, since the characteristic polynomial of the recurrence in (1.1), namely \(x^k - x^{k-1} - \ldots - x - 1\), has just one root \(\alpha\) such that \(|\alpha| > 1\) (see for instance \([39]\)), hence \(F^{(k)} \cap R\) is a finite set, for all \(k \geq 2\). See also the article \([32]\) for some results on the set \(F^{(k)} \cap \mathbb{P}\) and a conjecture on the intersection \(F^{(k)} \cap F^{(m)}\). We point out that this last intersection is, to the best of our knowledge, not known even in the easiest case \((k, m) = (2, 3)\), that is, for numbers that are both Fibonacci and Tribonacci. A possible way to find this intersection is to look at the Fibonacci and Tribonacci sequences modulo \(p^t\), where \(p\) is a prime number. We refer the reader to \([5], [13], [16], [17]\) for results of this nature. However, this approach seems to be hard to work in practice. This observation prompted the author to look for a more interesting and constructive approach which could be useful in the general case.

It is important to notice that Mignotte (see \([31]\)) showed that if \((u_n)_{n \geq 0}\) and \((v_n)_{n \geq 0}\) are two linearly recurrence sequences then, under some weak technical as-
sumptions, the equation

\[ u_n = v_m \]

has only finitely many solutions in positive integers \( m, n \). Moreover, all such solutions are effectively computable. Therefore, it seems reasonable to think that \( F^{(k)} \cap F^{(m)} \) is a finite set for all \( k \neq m \).

The goal of this paper is to apply transcendental tools to provide a method for studying the intersection \( F^{(k)} \cap F^{(m)} \), for integers \( 2 \leq k < m \) and determine completely this set for \((k, m) = (2, 3)\) (confirming the expectation). More precisely, our result is the following.

**Theorem 1.** The only solution of the Diophantine equation

\[ F_n = T_m \]

in positive integer numbers \( m \) and \( n \) with \( n > 3 \), is \((n, m) = (7, 6)\). Hence, \( F \cap T = \{0, 1, 2, 13\} \).

We organize this paper as follows. In Section 2, we will recall some useful properties such as a result of Matveev on linear forms in three logarithms and the reduction method of Baker-Davenport that we will use in the proof of Theorem 1. In Section 3, we first use Baker’s method to obtain a bound for \( n \), then we completely prove Theorem 1 by means of the Baker-Davenport reduction method.

2. **Auxiliary results**

We recall the well-known Binet’s formula:

\[ F_n = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}} \quad \text{for all} \ n \geq 0, \]

where \( \varphi = (1 + \sqrt{5})/2 \). It is almost unnecessary to stress that this is a very helpful formula which moreover allows to deduce that

\[ \varphi^{n-2} < F_n < \varphi^{n-1} \quad \text{for all} \ n \geq 1. \]

In 1982, Spickerman [36] found the following “Binet-style” formula for the Tribonacci sequence:

\[ T_n = \frac{\alpha^n}{-\alpha^2 + 4\alpha - 1} + \frac{\beta^n}{-\beta^2 + 4\beta - 1} + \frac{\gamma^n}{-\gamma^2 + 4\gamma - 1} \quad \text{for all} \ n \geq 0, \]
where \( \alpha, \beta, \gamma \) are the roots of \( x^3 - x^2 - x - 1 = 0 \). Explicitly, we have

\[
\alpha = \frac{1}{3} + \frac{1}{3}(19 - 3\sqrt{33})^{1/3} + \frac{1}{3}(19 + 3\sqrt{33})^{1/3},
\]

\[
\beta = \frac{1}{3} - \frac{1}{6}(1 + i\sqrt{3})(19 - 3\sqrt{33})^{1/3} - \frac{1}{6}(1 - i\sqrt{3})(19 + 3\sqrt{33})^{1/3},
\]

\[
\gamma = \frac{1}{3} - \frac{1}{6}(1 - i\sqrt{3})(19 - 3\sqrt{33})^{1/3} - \frac{1}{6}(1 + i\sqrt{3})(19 + 3\sqrt{33})^{1/3}.
\]

Another interesting formula due to Spickermann is

\[
T_n = \text{Round} \left[ \frac{\alpha}{(\alpha - \beta)(\alpha - \gamma)} \alpha^n \right],
\]

where, as usual, \( \text{Round}[x] \) is the nearest integer to \( x \).

Since \( \alpha^{-2} < \alpha/(\alpha - \beta)(\alpha - \gamma) = 0.33622 \ldots < \alpha \), the above identity yields the bounds

\[
\alpha^{n-3} < T_n < \alpha^{n+2} \quad \text{for all } n \geq 1.
\]

The Fibonacci and Tribonacci numbers can also be computed using the generating functions

\[
\frac{z}{1 - z - z^2} = 1 + z + 2z^2 + 3z^3 + 5z^4 + 8z^5 + 13z^6 + 21z^7 + 34z^8 + \ldots, \tag{2.3}
\]

\[
\frac{z}{1 - z - z^2 - z^3} = 1 + z + 2z^2 + 4z^3 + 7z^4 + 13z^5 + 24z^6 + 44z^7 + 81z^8 + \ldots. \tag{2.4}
\]

In order to prove Theorem 1, we will use a lower bound for a linear form in three logarithms à la Baker and such a bound was given by the following result of Matveev [30].

**Lemma 1.** Let \( \alpha_1, \alpha_2, \alpha_3 \) be real algebraic numbers and let \( b_1, b_2, b_3 \) be nonzero rational numbers. Define

\[
\Lambda = b_1 \log \alpha_1 + b_2 \log \alpha_2 + b_3 \log \alpha_3.
\]

Let \( D \) be the degree of the number field \( \mathbb{Q}(\alpha_1, \alpha_2, \alpha_3) \) over \( \mathbb{Q} \) and let \( A_1, A_2, A_3 \) be positive real numbers which satisfy

\[
A_j \geq \max\{Dh(\alpha_j), |\log \alpha_j|, 0.16\} \quad \text{for } j = 1, 2, 3.
\]

Assume that

\[
B \geq \max\{1, \max\{|b_j|A_j/A_1; 1 \leq j \leq 3\}\}.
\]
Define also
\[ C = 6750000 \cdot e^4(20.2 + \log(3.5 D^2 \log(eD))). \]

If $\Lambda \neq 0$, then
\[ \log |\Lambda| \geq -CD^2 A_1 A_2 A_3 \log(1.5 e D B \log(eD)). \]

As usual, in the above statement, the logarithmic height of an $s$-degree algebraic number $\alpha$ is defined as
\[ h(\alpha) = \frac{1}{s} \left( \log |a| + \sum_{j=1}^{s} \log \max\{1, |\alpha(j)|\} \right), \]
where $a$ is the leading coefficient of the minimal polynomial of $\alpha$ (over $\mathbb{Z}$), $(\alpha(j))_{1 \leq j \leq s}$ are the conjugates of $\alpha$ and, as usual, the absolute value of the complex number $z = a + bi$ is $|z| = \sqrt{a^2 + b^2}$.

After finding an upper bound on $n$ which is generally too large, the next step is to reduce it. For that, we need a variant of the famous Baker-Davenport lemma, which is due to Dujella and Pethö [6]. For a real number $x$, we use $\|x\| = \min\{|x - n|: n \in \mathbb{N}\} = |x - \text{Round}[x]|$ for the distance from $x$ to the nearest integer.

**Lemma 2.** Suppose that $M$ is a positive integer. Let $p/q$ be a convergent of the continued fraction expansion of $\gamma$ such that $q > 6M$ and let $\| \mu q \| - M \| \gamma q \|$, where $\mu$ is a real number. If $\varepsilon > 0$, then there is no solution to the inequality
\[ 0 < m\gamma - n + \mu < AB^{-m} \]
in positive integers $m, n$ with
\[ \frac{\log(Aq/\varepsilon)}{\log B} \leq m < M. \]

See Lemma 5, a) in [6]. Now, we are ready to deal with the proofs of our results.
3. The proof of Theorem 1

3.1. Finding a bound on \( n \). By Binet’s formulae (2.1) and (2.2) we get

\[
\frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}} = \frac{\alpha^m}{\alpha'} + \frac{\beta^m}{\beta'} + \frac{\gamma^m}{\gamma'}.
\]

Let us denote by \( \alpha', \beta', \gamma' \) the values of \( Q(x) = -x^2 + 4x - 1 \) at \( x = \alpha, \beta, \gamma \), respectively. By (2.2) and equation (1.2), we have

\[
\frac{\varphi^n}{\sqrt{5}} - \frac{\alpha^m}{\alpha'} = \frac{(-1)^n \varphi^{-n}}{\sqrt{5}} + \frac{\beta^m}{\beta'} + \frac{\gamma^m}{\gamma'}, \quad m,n \geq 1.
\]

More precisely,

\[
(3.1) \quad \left| \frac{\varphi^n}{\sqrt{5}} - \frac{\alpha^m}{\alpha'} \right| \leq \left| \frac{\varphi^{-1}}{\sqrt{5}} \right| + 2 \left| \frac{\beta}{\beta'} \right| < 0.67 \quad \text{for any } m,n \geq 1
\]

where in the last inequality we have used \( |\beta| = |\gamma| = 0.73735 \ldots \) and \( |\beta'| = |\gamma'| = 3.84631 \ldots \).

Define

\[
\Lambda = \Lambda(m,n) = m \log \alpha - n \log \varphi + \log \left( \frac{\sqrt{5}}{\alpha'} \right).
\]

Then

\[
\Lambda = \log \left( \frac{\alpha^m \varphi^{-n} \sqrt{5}}{\alpha'} \right),
\]

which yields

\[
|e^\Lambda - 1| = \left| \frac{\alpha^m \varphi^{-n} \sqrt{5}}{\alpha'} - 1 \right|.
\]

On the other hand, from (3.1) we get

\[
\left| \varphi^n - \frac{\alpha^m \sqrt{5}}{\alpha'} \right| < 0.67 \cdot \sqrt{5} < 1.5.
\]

Hence

\[
|e^\Lambda - 1| = \frac{1}{\varphi^n} \left| \varphi^n - \frac{\alpha^m \sqrt{5}}{\alpha'} \right| < \frac{1.5}{\varphi^n}.
\]

Since \( \varphi = 1.61803 \ldots \), we have \( 1.5/\varphi^n < \varphi^{-n+1} \) and then

\[
(3.2) \quad |e^\Lambda - 1| < \varphi^{-n+1}.
\]
We claim that $\Lambda \neq 0$. In fact, towards a contradiction, suppose that $\Lambda = 0$ and thus $\alpha^m \sqrt{5}/\alpha' = \varphi^n$. Therefore $\alpha^{2m}/\alpha'^2$ is a quadratic algebraic number. However $\alpha^{2m}/\alpha'^2 \in \mathbb{Q}(\alpha)$ which is absurd, because $\alpha$ is a 3-degree algebraic number.

If $\Lambda > 0$, then $\Lambda < e^{\Lambda} - 1 < \varphi^{-n+2}$ (see (3.2)). If $\Lambda < 0$, then $1 - e^{-|\Lambda|} = |e^\Lambda - 1| < \varphi^{-n+2}$. Thus, for $\Lambda < 0$, we get

$$|\Lambda| < e^{|\Lambda|} - 1 < \frac{\varphi^{-n+1}}{1 - \varphi^{-n+1}} < \varphi^{-n+2},$$

where we have used the fact that $1 - \varphi^{-n+1} > 1/\varphi$ for all $n > 3$.

Hence, we have $|\Lambda| < \varphi^{-n+2}$ for any $\Lambda \neq 0$, which yields

(3.3) \quad \log |\Lambda| < -(n - 2) \log \varphi.

Now, we will apply Lemma 1. Take

$$\alpha_1 = \alpha, \quad \alpha_2 = \varphi, \quad \alpha_3 = \sqrt{5}/\alpha', \quad b_1 = m, \quad b_2 = -n, \quad b_3 = 1.$$ 

Then $\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3) = \mathbb{Q}(\alpha, \varphi)$, $D = 6$ and $C < 1.2 \cdot 10^{10}$.

It is easy to verify that $1/\alpha'$ is a root of $44x^3 - 2x - 1$ and that $\sqrt{5}/\alpha'$ is a root of $1936x^6 - 880x^4 + 100x^2 - 125$. Since $\sqrt{5}/\alpha'$ is a 6-degree algebraic number, its minimal polynomial (over $\mathbb{Z}$) is $1936x^6 - 880x^4 + 100x^2 - 125$. Using direct calculation, we verify that the absolute value of every root of the minimal polynomial is less than 1. Hence $h(\alpha_3) < (\log 1936)/6 < 1.262$. Next, we have $h(\alpha_1) = (\log \alpha)/3 = 0.204$ and $h(\alpha_2) = (\log \varphi)/2 < 0.241$. We then take $A_1 = 1.22$, $A_2 = 1.45$ and $A_3 = 7.58$. Since (1.2) implies $n > m$, we have

$$\max \{1, \max\{|b_j|A_j/A_1; \ 1 \leq j \leq 3\}\} = \max\{m, 1.2n\} = 1.2n =: B.$$ 

Hence, Lemma 1 yields

(3.4) \quad \log |\Lambda| > -6.8 \cdot 10^{12} \log(82n).

Combining the estimates (3.3) and (3.4), we get

$$6.8 \cdot 10^{12} \log(82n) > (n - 2) \log \varphi,$$

and this inequality implies $n < 6 \cdot 10^{14}$ and, by the trivial estimate $m < n$, we have $m < 6 \cdot 10^{14}$. In order to improve the estimates, we use the bounds on $F_n$ and $T_m$ together with Equation (1.2) to obtain $\alpha^{m-3} < T_m = F_n < \varphi^{n-1}$, which yields
\[ m < 0.8n + 2.2. \] Hence, \( m < 4.8 \cdot 10^{14} \). Similarly, \( \varphi^{n-2} < F_n = T_m < \alpha^{m+2} \) yields \( n < 1.3m + 4.6. \)

**3.2. Reducing the bound.** The next goal is to reduce the bound on \( m \). For that, let us suppose, without loss of generality, that \( \Lambda > 0 \) (the other case can be handled in a similar way by considering \( 0 < \Lambda' = -\Lambda \)).

We know that \( 0 < \Lambda < \varphi^{-n+2} \) and therefore

\[
0 < m \log \alpha - n \log \varphi + \log \left( \frac{\sqrt{5}}{\alpha'} \right) < \varphi^{-m+2}.
\]

Dividing by \( \log \varphi \), we get

\[
(3.5) \quad 0 < m \hat{\gamma} - n + \mu < 5.45 \cdot \varphi^{-m},
\]

with \( \hat{\gamma} = \log \alpha / \log \varphi \) and \( \mu = \log(\sqrt{5}/\alpha') / \log \varphi \).

Surely \( \hat{\gamma} \) is an irrational number (actually, this number is transcendental by the Gelfond-Schneider theorem: if \( \alpha \) and \( \beta \) are algebraic numbers with \( \alpha \neq 0 \) or \( 1 \), and \( \beta \) is irrational, then \( \alpha^\beta \) is transcendental). So, let us denote by \( p_n/q_n \) the \( n \)th convergent of its continued fraction.

In order to reduce our bound on \( m \), we will use Lemma 2. For that, taking \( M = 4.8 \cdot 10^{14} \), we have that

\[
\frac{p_{33}}{q_{33}} = \frac{53739149317980067}{42436582738078750}.
\]

and then \( q_{33} > 6M \). Moreover, we get

\[
\| \mu q_{33} \| - M \| \hat{\gamma} q_{33} \| > 0.028 =: \varepsilon.
\]

Thus all the hypotheses of Lemma 2 are satisfied and we take \( A = 5.45 \) and \( B = \varphi \). It follows from Lemma 2 that there is no solution of the inequality in (3.5) (and then for the Diophantine equation (1.2)) in the range

\[
\left[ \left\lfloor \frac{\log(Aq_{33}/\varepsilon)}{\log B} \right\rfloor + 1, M \right] = [91, 4.8 \cdot 10^{14}].
\]

Therefore \( m \leq 90 \) and then \( n \leq 120 \). To conclude, we use the formulas in (2.3) and (2.4) together with the Mathematica command

\[
\text{Intersection[CoefficientList[Series[x/(1-x-x^2), x, 0, 120], x]}, \text{CoefficientList[Series[x/(1-x-x^2-x^3), x, 0, 90], x]}\]
\]

to find the possible solutions. Fastly, Mathematica returns us the set \( \{0, 1, 2, 13\} \) as its answer. This completes the proof.
4. Final remarks and a conjecture

We point out that the method in proof of Theorem 1 is quite general and that it can be used to work on the intersection of two arbitrary \( k \)-generalized Fibonacci sequences. In fact, in a similar fashion, we found the set \( F^{(k)} \cap F^{(m)} \) for \( 4 \leq k < m \leq 10 \). These cases suggest that the following statement (which is Conjecture 1 in [32]) should be true.

**Conjecture 1.** Let \( k < m \) be positive integer numbers. Then

\[
F^{(k)} \cap F^{(m)} = \begin{cases} 
\{0, 1, 2, 13\}, & \text{if } (k, m) = (2, 3), \\
\{0, 1, 2, 4, 504\}, & \text{if } (k, m) = (3, 7), \\
\{0, 1, 2, 8\}, & \text{if } k = 2 \text{ and } m > 3, \\
\{0, 1, 2, \ldots, 2^{k-1}\}, & \text{otherwise}. 
\end{cases}
\]

When working on these cases it may be helpful that the polynomials \( \psi_k(x) := x^k - x^{k-1} - \ldots - x - 1 \) are irreducible over \( \mathbb{Q}[x] \) with just one zero outside the unit circle. That single zero is located between \( 2(1 - 2^{-k}) \) and 2 (as seen in [39]). Also, in a recent paper, G. Dresden [7, Theorem 1] gave a simplified “Binet-like” formula for \( F_n^{(k)} \):

\[
F_n^{(k)} = \sum_{i=1}^{k} \frac{\alpha_i - 1}{2 + (k + 1)(\alpha_i - 2)} \alpha_i^{n-1}
\]

for \( \alpha_1, \ldots, \alpha_k \) being the roots of \( \psi_k(x) = 0 \). There are many other ways of representing these \( k \)-generalized Fibonacci numbers, as can be seen in [10], [11], [12], [14], [15]. Also, it was proved in [7, Theorem 2] that

\[
F_n^{(k)} = \text{Round} \left[ \frac{\alpha - 1}{2 + (k + 1)(\alpha - 2)} \alpha^{n-1} \right],
\]

where \( \alpha \) is the dominant root of \( \psi_k(x) \).

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