Abstract. In this note, we shall prove the existence of an uncountable sub-
set of Liouville numbers (which we call the set of ultra-Liouville numbers) for
which there exists uncountably many transcendental analytic functions map-
ping the subset into itself.

1. Introduction

A real number \( \xi \) is called a Liouville number, if there exists a rational sequence

\[(p_k/q_k)_{k \geq 1}, \text{ with } q_k > 1, \text{ such that} \]

\[0 < \left| \xi - \frac{p_k}{q_k} \right| < q_k^{-k}, \text{ for } k = 1, 2, \ldots.\]

The set of the Liouville numbers is denoted by \( L \).

The name arises because Liouville [4] in 1844 showed that all Liouville numbers
are transcendental, establishing thus the first explicit examples of transcendental
numbers. The number \( \ell := \sum_{n \geq 1} 10^{-n!} \), the so-called Liouville constant, is a
standard example of a Liouville number. In 1962, Erdős [3] proved that every real
number can be written as the sum and (if it is non zero) the product of two Liouville
numbers, as a consequence of the fact that \( L \) is a rather large set in a topological
sense: it is a dense \( G_\delta \) set.

In his pioneering book, Maillet [6, Chapitre III] discusses some arithmetic prop-
erties of Liouville numbers. One of them is that, given a rational function \( f \), with
rational coefficients, if \( \xi \) is a Liouville number, then so is \( f(\xi) \). We observe that
the converse of this result is not valid in general, e.g., taking \( f(x) = x^2 \), then
\( \zeta := \sqrt{3 + \ell}/4 \) is not a Liouville number [1, Theorem 7.4], but \( f(\zeta) \) is. Also
the rational coefficients cannot be taken algebraic (with at least one of them non-
trivial). For instance, \( \ell \sqrt{3/2} \) is not a Liouville number, see [6, Théorème 13]. In
fact, \( \ell \sqrt{3/2} \) is a \( U_2 \)-number (for the definition of a \( U_2 \)-number and this result, see
[2]).

An algebraic function is a function \( f(x) \) which satisfies \( P(x, f(x)) = 0 \), where
\( P(x, y) \) is a polynomial with complex coefficients. For instance, functions that
can be constructed using only a finite number of elementary operations are exam-
pies of algebraic functions. A function which is not algebraic is, by definition, a
transcendental function. Common examples are the trigonometric functions, the
exponential function, and their inverses.

In 1984, in one of his last papers, K. Mahler [5] stated several questions for which,
according to him, ‘perhaps further research might lead to interesting results’. His

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first question is related to Liouville numbers. In particular, this question asks the following:

**Question.** Are there transcendental entire functions \( f(z) \) such that if \( \xi \) is any Liouville number, then so is \( f(\xi) \)?

He also said that: ‘The difficulty of this problem lies of course in the fact that the set of all Liouville numbers is non-enumerable’.

The study of similar problems has attracted the attention of several mathematicians. Let \( A \) and \( B \) be subsets of \( \mathbb{C} \) with \( A \subseteq B \) and let \( \Sigma_A(B) \) be the set of all transcendental analytic functions \( f : B \to B \) such that \( f(A) \subseteq A \). In 1886, Weierstrass proved that the set \( \Sigma_\mathbb{Q}(\mathbb{R}) \) has the power of continuum. Moreover, he asserted that \( \Sigma_\mathbb{C}(\mathbb{C}) \neq \emptyset \). In 1896, Stäckel [7] confirmed the Weierstrass assertion by proving that for each countable subset \( \Sigma \subseteq \mathbb{C} \) and each dense subset \( T \subseteq \mathbb{C} \), there is a transcendental entire function \( f \) such that \( f(\Sigma) \subseteq T \). In particular, if \( A \) is a countable dense subset of \( \mathbb{C} \), then \( \Sigma_A(\mathbb{C}) \) is uncountable. Consult the very extensive annotated bibliography of [8] for additional references and history. Note that the Mahler question can be rephrased as: is \( \Sigma_L(\mathbb{C}) \neq \emptyset \)?

Set, inductively, \( \exp^n(x) = \exp(\exp^{n-1}(x)) \) and \( \exp^0(x) = x \). Now, let us define the following class of numbers:

**Definition.** A real number \( \xi \) is called an ultra-Liouville number, if for every positive integer \( k \), there exist infinitely many rational numbers \( p/q \), with \( q > 1 \), such that

\[
0 < \left| \xi - \frac{p}{q} \right| < \frac{1}{\exp^k(q)}.
\]

The set of the ultra-Liouville numbers will be denoted by \( L_{\text{ultra}} \).

It follows from the definition that \( L_{\text{ultra}} \subseteq L \) is also a dense \( G_δ \) set (in particular it is uncountable) which means that \( L_{\text{ultra}} \) is a large set in a topological sense. In particular, every real number can be written as the sum and (if it is not zero) the product of two ultra-Liouville numbers, as in [3]. However, from a metrical point of view, both sets \( L \) and \( L_{\text{ultra}} \) are very small: they have Hausdorff dimension zero.

The aim of this paper is to investigate a problem related to Mahler’s question concerning \( L_{\text{ultra}} \). More precisely, our main result is the following

**Theorem 1.** The set \( \Sigma_{L_{\text{ultra}}}(\mathbb{C}) \) is uncountable.

In order to prove that, we shall prove a stronger result about the behavior of some functions in \( \Sigma_\mathbb{Q}(\mathbb{C}) \). For a rational number \( z \), we denote by \( \text{den}(z) \) its denominator. We prove that

**Theorem 2.** There exist uncountably many functions \( f \in \Sigma_\mathbb{Q}(\mathbb{C}) \) with \( 1/2 < f'(x) < 3/2, \forall x \in \mathbb{R} \), such that

\[
(*) \quad \text{den}(f(p/q)) < q^{8q^2},
\]

for all \( p/q \in \mathbb{Q} \), with \( q > 1 \). In particular, \( \text{den}(f(p/q)) < e^{e^q} - 1 \), if \( q \geq 7 \).

2. The proofs

2.1. Proof that Theorem 2 implies Theorem 1. Given an ultra-Liouville number \( \xi \) and a positive integer \( k \), there exist infinitely many rational numbers \( p/q \) with
q ≥ 7 and such that

\[ 0 < \left| \xi - \frac{p}{q} \right| < \frac{1}{\exp(k+2)}(q). \]

Let \( f \) be a function as in Theorem 2. By the Mean Value Theorem, we obtain

\[ \left| f(\xi) - f \left( \frac{p}{q} \right) \right| \leq \frac{3}{2} \left| \xi - \frac{p}{q} \right| < \frac{3}{2} \exp(k+2)(q). \]

We know that \( f(p/q) = a/b \), with \( b < e^x - 1 \). Then \( \frac{3}{2} \exp(k)b < \exp(k+2)(q) \) and hence

\[ \left| f(\xi) - \frac{a}{b} \right| = \left| f(\xi) - f \left( \frac{p}{q} \right) \right| < \frac{1}{\exp(b)}. \]

This implies that \( f(\xi) \) is an ultra-Liouville number as desired. \( \square \)

2.2. Proof of Theorem 2. Before starting the proof, we shall state three useful facts (which can be easily proved)

- For any distinct \( y, b \in [-1, 1] \), we have \( |\sin(y - b)| > |y - b|/3 \).
  (Indeed, the function \( \sin(x)/x \) is decreasing for \( x \in (0, \pi] \), and \( \sin(2)/2 > 1/3 \)).
- For any distinct \( x, y \in \mathbb{Q} \cap [0, 1/2] \), with \( \text{den}(x), \text{den}(y) \leq n \), we have
  \[ |\cos(2\pi x) - \cos(2\pi y)| \geq \frac{4}{n^3}. \]
  (Indeed, we can assume \( x < y \); we can also assume \( y \leq 1/4 \): if \( 1/4 \leq x \leq 1/2 \), we use that \( |\cos(2\pi x) - \cos(2\pi y)| = |\cos(2\pi(1/2 - x)) - \cos(2\pi(1/2 - y))| \), and, if \( x < 1/4 < y \) we use that \( |\cos(2\pi x) - \cos(2\pi y)| > |\cos(2\pi x) - \cos(2\pi \cdot 1/4)| > 1 - 4x \geq 1/n \geq 4/n^3 \), since \( \text{den}(y) \geq 2 \); now we have two cases: if \( x = 0 \) then \( \cos(2\pi x) = 1 - \cos(2\pi y) = 2\sin^2(\pi y) \geq 8/n^2 \geq 4/n^3 \); and, if \( 0 < x < y \) then \( x > 1/n \) and, by the mean value theorem, \( |\cos(2\pi x) - \cos(2\pi y)| \geq 2\pi \sin(2\pi \xi)(2\pi y - 2\pi x) \geq 8\pi(x - y) \geq 8\pi(y - x)/n \geq 8\pi/n^3 > 4/n^3 \).
- For every \( \epsilon \in (0, 2] \), any interval of length \( \epsilon \) contains at least two rational numbers with denominator \( \leq [2/\epsilon] \). Indeed, if \( m = [2/\epsilon] \) and \( (a, b) \) is the interior of the interval, we have \( b - a > \epsilon \geq 2/m \), and so, for \( k = [ma] + 1 \), we have \( ma < k \leq ma + 1 \), and so \( ma < k < k + 1 \leq ma + 2 < ma + m(b - a) = nb \), which implies \( a < k/m < (k + 1)/m < b \).

Consider the following enumeration of \( \mathbb{Q} \cap [0, 1/2] \):

\[ \{x_1, x_2, \ldots \} = \left\{ \frac{0}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{2}{5}, \frac{1}{6}, \ldots \right\}, \]

where we consider only irreducible fractions ordered in the following way: \( x_1 = 0/1 \); for every \( k \geq 1 \), if \( x_k = p/q \) with \( 2p < q - 2 \) then \( x_{k+1} = r/q \) where \( r \) is the minimum with \( p < r \leq q/2 \) and \( \gcd(r, q) = 1 \), and if \( 2p \geq q - 2 \) then \( x_{k+1} = 1/(q + 1) \). The set \( A = \mathbb{Q} \cap [0, 1/2] \) has the properties that \( \cos(2\pi x) \neq \cos(2\pi y) \) for every \( x \neq y \) in \( A \), and that for every \( z \in \mathbb{Q} \) there is (exactly one) \( x \in A \) with \( \cos(2\pi x) = \cos(2\pi z) \).

One can see that \( \text{den}(x_n) \geq \sqrt{n} \), for all \( n \geq 1 \); indeed, the number of positive integers \( n \) for which the denominator of \( x_n \) is equal to \( k \) is at most \( k \) for every \( k \geq 1 \), so the maximum positive integer \( n \) for which the denominator of \( x_n \) is at most \( k \) is at most \( 1 + 2 + \cdots + k = k(k + 1)/2 \leq k^2 \).

Define \( B_n = \{y_1, y_2, \ldots, y_n\} \) with \( y_k := \cos(2\pi x_k) \) and define \( f \) by

\[ f(x) = x + g(\cos(2\pi x)), \]
where \( g(y) = \sum_{n=1}^{\infty} c_n g_n(y) \) and \( g_n(y) = \prod_{k \in B_n} \sin(y - b) \). Note that \( f(x + 1) = f(x) + 1 \) and so it is enough to consider \( \mathbb{Q} \cap [0,1] \) in order to characterize \( f \) on \( \mathbb{Q} \). Notice also that, in order to show that \( f(x) \in \mathbb{Q} \) for every \( x \in \mathbb{Q} \), it is enough to prove this for \( x \in \mathbb{A} \). Indeed, given \( z \in \mathbb{Q} \), take \( x \in \mathbb{A} \) with \( \cos(2\pi x) = \cos(2\pi z) \).

Then we have \( f(z) - z = g(\cos(2\pi z)) = g(\cos(2\pi x)) = f(x) - x \), and so, if \( f(x) \in \mathbb{Q} \), then \( f(z) = f(x) + z - x \in \mathbb{Q} \); in particular, if \( z \in \mathbb{Z} \) then \( f(z) = z \), since \( f(0) = 0 \).

Now, we shall choose inductively the constants \( c_n \) so that \( f \) will satisfy the desired conditions in Theorem 2. The first requirements are \( c_n = 0 \) for \( 1 \leq n \leq 5 \) and \( |c_n| < 1/n^n \) for every positive integer \( n \). On the other hand, for all \( y \) belonging to the open ball \( B(0, R) \) one has that

\[
|g_n(y)| < \prod_{b \in B_n} e^{|y-b|} \leq e^{n(R+1)},
\]

where we used the fact that \( b \in [-1,1] \). Thus, since \( |c_n| < 1/n^n \), we get \( |c_n g_n(y)| \leq (e^{R+1}/n)^n \) from which \( g \) (and so \( f \)) is an entire function, since the series \( g(y) = \sum_{n=1}^{\infty} c_n g_n(y) \), which defines \( g \), converges uniformly in any of these balls. Moreover, for \( x \in \mathbb{R} \), we have \( |g(x)| \leq n \), and so \( f'(x) = 1 - 2\pi \sin(2\pi x) \sum_{n=1}^{\infty} c_n g_n(\cos(2\pi x)) \in (1/2,3/2) \), since \( \sum_{n=0}^{\infty} n/n^n < 1/4\pi \).

Suppose that \( c_1,\ldots,c_{n-1} \) have been chosen such that \( f(x_1),\ldots,f(x_n) \) have the desired property (notice that the choice of \( c_1,\ldots,c_{n-1} \) determines the values of \( f(x_1),\ldots,f(x_n) \), independently of the values of \( c_k,k \geq n \); in particular, since \( c_k = 0 \) for \( 1 < k \leq 5 \), we have \( f(x_k) = x_k \) for \( 1 \leq n \leq 6 \)). Now, we shall choose \( c_n \) for which \( f(x_{n+1}) \) satisfies the requirements.

Let \( t \leq n \) be positive integers with \( n \geq 5 \). Then \( \text{den}(x_{n+1}),\text{den}(x_1) \leq n \) (indeed, \( \text{den}(x_{n+1}) = 5 \) and \( \text{den}(x_{n+1}) - \text{den}(x_1) \leq 1, \forall n \geq 1 \)). Since \( \cos(2\pi x_{n+1}) \neq \cos(2\pi x_1) \), then \( |y_n - y_1| \geq 4/n^3 \). Therefore

\[
|\sin(y_{n+1} - y_1)| > \frac{|y_{n+1} - y_1|}{3} > \frac{4}{3n^3} > \frac{1}{n^3}
\]

yielding \( |g_n(y_{n+1})| > n^{-3n} \). Thus \( c_n g_n(y_{n+1}) \) runs through an interval of length larger than \( 2/n^4n \). Now, we may choose (in at least two ways) \( c_n \) such that \( g(y_{n+1}) \) is a rational number with denominator at most \( n^{4n} \).

Given \( z \in \mathbb{Q} \), let \( q = \text{den}(z) \); if \( q = 1 \) then \( z \in \mathbb{Z} \) and so \( f(z) = z \) and thus \( \text{den}(f(z)) = 1 \leq q^8q^2 \).

Otherwise, \( q > 1 \), and there is a positive integer \( k \) with \( \cos(2\pi x_k) = \cos(2\pi z) \), so \( x_k \in \mathbb{Q} \) and \( z \) have the same denominator; indeed, in this case, we have \( z - x_k \in \mathbb{Z} \) or \( z + x_k \in \mathbb{Z} \). Thus \( \text{den}(f(z) - z) = \text{den}(g(\cos(2\pi x_k)) = \text{den}(g(\cos(2\pi x_k))) \leq (k-1)^4(k-1) < k^{4(k-1)} \). Since \( q = \text{den}(z) = \text{den}(c_k) \geq \sqrt{k} \), we get \( \text{den}(f(z) - z) \leq k^{4(k-1)} \leq (q^2)^{4(q^2-1)} = q^8(q^2-1) \). Then we have

\[
\text{den}(f(z)) \leq \text{den}(z) \text{den}(f(z) - z) = q \text{den}(f(z) - z) \leq q \cdot q^8(q^2-1) \leq q^{8q^2}
\]

as desired.

The proof that we can choose \( f \) to be transcendental follows because there is a binary tree of different possibilities for \( f \). (If we have chosen \( c_1, c_2, \ldots, c_{n-1} \), different choices of \( c_n \) give different values of \( f(y_{n+1}) \), which does not depend on the values of \( c_k \) for \( k > n \), and so different functions \( f_j \). Thus, we have constructed uncountably many possible functions, and the algebraic entire functions taking \( \mathbb{Q} \) into itself must be polynomials belonging to \( \mathbb{Q}[z] \), which is a countable subset.
In fact, we can prove that all functions constructed above are transcendental, unless $c_n = 0, \forall n \in \mathbb{N}$: if such a function $f$ is not transcendental, then $f$ would be polynomial, since it is an entire function. However, the property $f(x+1) = f(x) + 1$ would imply $f(x) = x + c$, for some $c > 0$. Then $g(\sin(2\pi x))$ is a constant, but this leads to a contradiction, since $g(y_1) = 0$ and $g(y_{k+1}) = c_k \prod_{b \in B_k} \sin(y_{k+1} - b) \neq 0$, where $k$ is minimal such that $c_k \neq 0$.

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