ON THE DIOPHANTINE EQUATION

\[ 3^{2n} - 2 \cdot 3^m + 1 = k^2 \]

Gervasio G. Bastos\textsuperscript{1,*} and Diego Marques\textsuperscript{2,†}

\textsuperscript{1}Departamento de Matemática, Universidade Estadual do Ceará, Ceará, Brazil
\textsuperscript{2}Departamento de Matemática, Universidade de Brasília, Brasília, Brazil

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Abstract

In 1981, F. Beukers used a hyper-geometric method for proving that the well-known generalized Ramanujan-Nagell equation

\[ x^2 + C = p^n, \ p \text{ prime}, \]

has at most one solution in positive integers \( x \) and \( n \), where \( C \) and \( p \) are previously fixed, with a few exceptions.

In this note, we give an elementary proof of this result when \( n \) is even as well as the complete solution of such an equation when \( C \) is a power of 2009. Moreover, we prove that the previous result is surprisingly connected with the title equation which allows us to find all solutions for that equation.

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1 Introduction

The Diophantine equation

\[ x^2 + C = y^n, \ x \geq 1, \ y \geq 1, \ n \geq 3 \] (1)

has a rich history and it has attracted the attention of several mathematicians. Several papers have been written on this topic, specially for particular values of \( C \). The first non-trivial result is due to Lebesgue [21] and date back to the 1850.

\*ggbastos@ufc.br
\†diego@mat.unb.br
He proved that the above equation has no solutions for $C = 1$. In 1965, Ko [18] proved that if $C = -1$, then the only solution is $(x, y, n) = (3, 2, 3)$. In 2004, Tengely [31] solved the above equation with $C = B^2$ and $B \in \{3, 4, \ldots, 501\}$. The case when $C = p^k$, where $p$ is a prime number, was studied for $p = 2$, in [8, 19, 20] for $p = 3$ in [6, 7, 22], for $p = 5$ in [1, 2] and for $p = 7$ in [25]. Some advances on an arbitrary prime $p$ appear in [5]. The equations $x^2 + C = y^n$ with $1 \leq C \leq 100$ were completely solved in [12]. Also, the solutions when $x$ and $y$ are coprime $C = 2^a \cdot 3^b$, $C = 2^a \cdot 5^b$ and $C = 5^a \cdot 13^b$ can be found in [23, 24, 3], respectively. The more recent progress on the subject concerns to cases $C = 5^a \cdot 11^b$, $C = 2^a \cdot 11^b$, $C = 2^a \cdot 3^b \cdot 11^c$ and can be found in [14, 15, 16].

Also, several authors become interested in the equation (1) when the variable $y$ is replaced by a positive integer number. The equation

$$x^2 + C = t^n,$$

where $C$ and $t$ are given integers, is called the generalized Ramanujan-Nagell equation. For instance, there is quite an extensive literature concerning the equation

$$x^2 + C = p^n, \ p \text{ prime},$$

beginning for the case $C = 7$ and $p = 2$, which was posed in a work of Ramanujan [28, 29], in 1913 and first solved by Nagell [27] in 1948. The case $C = 11$ and $p = 3$ was treated by Cohen [13] in 1976. Consult its very extensive annotated bibliography for additional references and history. As a final remark, we point out that, in 1960, Apéry [4] showed that equation (2), when $p \nmid C$, has at most two solutions.

Here, we are particularly interested in solving the Diophantine equation

$$3^{2n} - 2 \cdot 3^n + 1 = k^2$$

We prove that the possible solutions for the above equation are related to the solubility of the generalized Ramanujan-Nagell equation for $t = 9$. Our first result is the following

**Theorem 1.** Let $C$ be a positive integer. Then the Diophantine equation

$$x^2 + C = 3^{2n}$$

has at most one solution in positive integers $x$ and $n$.

It is important to pay attention that Eq. (4) has solution only when $C \equiv 0, 2 \pmod{3}$.

After, we shall combine two powerful techniques in number theory, namely, the Baker’s theory on linear forms in logarithms and some tools from Diophantine approximation, due to Baker and Davenport to find a general method for solving the equation (4) for values of $C$ previously fixed. As application of it, we derive the following

**Theorem 2.** The Diophantine equation

$$x^2 + 2009^t = 3^{2n}$$

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has no solution in positive integers \(x, t\) and \(n\).

Finally, we prove

**Theorem 3.**  The only solutions of the Diophantine equation

\[3^{2n} - 2 \cdot 3^m + 1 = k^2\]

in positive integers \(m, n\) and \(k\), are those related to \(m = n\), i.e., \((n, m, k) = (n, n, 3^{n} - 1)\).

We point out that our method is quite general and can be applied by replacing \(3\) in the title equation by any odd prime number \(p\).

## 2 The Diophantine equation \(x^2 + C = 3^{2n}\)

### 2.1 The proof of Theorem 1

It is important to get noticed that Beukers [10, 11] proved that the equation (2) (and consequently Eq. (4)) has at most one solution except when \((p, C) = (3, 2)\) or \((4t^2 + 1, 3t^2 + 1)\), for a positive \(t\). In all these exceptional cases, the pair \((x, n) = (1, 1)\) is a direct solution and so Theorem 1 is according to Beukers result. He used refined techniques on hyper-geometric methods for proving these results.

Here we will present an elementary demonstration of the Theorem 1 which was discovered by Professor F. A. Germano who has communicated us his nice proof by e-mail.

**Proof.** Suppose that \(x, y, m, n\) are positive integer numbers such that \(x^2 + C = 3^{2m}\) and \(y^2 + C = 3^{2n}\). We shall show that \(m = n\) and consequently \(x = y\). First of all, we note that

\[(3^m + x)(3^m - x) = C = (3^n + y)(3^n - y)\]

Without losing any generality, we can suppose \(\gcd(C, 3) = \gcd(x, 3) = 1\). In fact, we have \(x = 3^u a, C = 3^v b\), where \(a, b \in \mathbb{N}, 3 \nmid ab, u\) and \(v\) are nonnegative integer numbers. Hence

\[x^2 + C = 3^{2u} a^2 + 3^v b = 3^{2m}\]

Of course, \(2m \geq \max\{2u, v\}\). Set \(\ell = \min\{2u, v\}\), we have \(\ell \leq 2u, v \leq 2m\) and \(3^{\ell}(3^{2u-\ell} a^2 + 3^{v-\ell} b) = 3^{2m}\). We then conclude that either \(2u = \ell = v\) or \(3 \nmid (3^{2u-\ell} a^2 + 3^{v-\ell} b)\). In the first case, we have

\[a^2 + b = 3^{2(m-u)}, \quad (6)\]

with \(m - u > 0\) and whence it is enough to prove the theorem for the equation (6). In the second case, we infer that \(1 = 3^{2u-\ell} a^2 + 3^{v-\ell} b > 1\) which is an absurd.
We have then $C = (3^m - x)(3^m + x) = r(2 \cdot 3^m - r)$, where $0 < r = 3^m - x < 3^m$. Thus, if $(x, m)$ is a solution of (4), we get an integer number $0 < r < 3^m$ such that $C = r(2 \cdot 3^m - r)$ and $3 \nmid r$. Therefore, for another solution $(y, n)$ of (4), there exists $0 < s = 3^n - y < 3^n$ such that $h = s(2 \cdot 3^n - s)$ and $3 \nmid s$.

We claim that $m = n$. Towards a contradiction, we may suppose $n > m$ (the other case can be handled in much the same way). This implies that $C = s(2 \cdot 3^n - s) = r(2 \cdot 3^m - r)$ and then $0 < s < r < 3^m$. Therefore, $r$ and $s$ have the same parity, since $3^m$ is even (keep in mind that $r$ and $s$ have the same parity). Thus, if $(x, m)$ is a solution of (4), we get an integer number $0 < s < r < 3^m$. Therefore, $r$ and $s$ have the same parity, since $s$ is even (keep in mind that $r$ and $s$ have the same parity). Thus $m = n$ as desired.

\section{The proof of Theorem 2}

\subsection{Auxiliary results}

Before proceeding further, we recall some results which will be very useful in what follows.

The main idea for proving the Theorem 2 is to use bounds à la Baker for a suitable linear form in three logarithms and then to deduce an upper bound on $t$. From the main result of Matveev [26], we extract the following result.

\textbf{Lemma 1.} Let $\alpha_1, \alpha_2, \alpha_3$ be real algebraic numbers and let $b_1, b_2, b_3$ be nonzero integer rational numbers. Define

$$A = b_1 \log \alpha_1 + b_2 \log \alpha_2 + b_3 \log \alpha_3$$

Let $D$ be the degree of the number field $\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3)$ over $\mathbb{Q}$ and let $A_1, A_2, A_3$ be real numbers which satisfy

$$A_j \geq \max\{D \log(\alpha_j), |\log \alpha_j|, 0.16\}, \text{ for } j = 1, 2, 3.$$

Assume that

$$B \geq \max\{1, \max\{|b_j| A_j/A_1; 1 \leq j \leq 3\}\}.$$ 

Define also

$$C_1 = 6750000 \cdot e^{4(20.2 + \log(3^{5.5}D^2 \log(eD)))}.$$

If $\Lambda \neq 0$, then

$$\log |\Lambda| \geq -C_1 D^2 A_1 A_2 A_3 \log(1.5eDB \log(eD)).$$

As usual, in the previous statement, the \textit{logarithmic height} of an $s$-degree algebraic number $\alpha$ is defined as

$$h(\alpha) = \frac{1}{s}(\log |\alpha| + \sum_{j=1}^{s} \log \max\{1, |\alpha^{(j)}|\}),$$
where $a$ is the leading coefficient of the minimal polynomial of $\alpha$ (over $\mathbb{Z}$) and $(\alpha^{(j)})_{1 \leq j \leq s}$ are the conjugates of $\alpha$.

After finding an upper bound on $t$ which is general too large, the next step is to reduce it. For this purpose, we need a variant of the famous Baker-Davenport lemma, which is due to Dujella and Pethö [17]. For a real number $x$, we use $\| x \| = \min\{|x - n| : n \in \mathbb{N}\}$ for the distance from $x$ to the nearest integer.

**Lemma 2.** Suppose that $M$ is a positive integer. Let $p/q$ be a convergent of the continued fraction expansion of the irrational number $\gamma$ such that $q > 6M$ and let $\epsilon = \| \mu q - M \| \gamma q \|$, where $\mu$ is a real number. If $\epsilon > 0$, then there is no solution to the inequality

$$0 < m\gamma - n + \mu < A \cdot B^{-m}$$

in positive integers $m, n$ with

$$\frac{\log(Aq/\epsilon)}{\log B} \leq m < M.$$

See Lemma 5, a.) in [17].

Now, we are ready to deal with the proof of our result.

### 2.2.2 The proof

**Finding a bound on $k$**

First, note that $t$ in the equation (5) must be odd, say $2k + 1$, because $x^2 \equiv 0, 1 \pmod{3}$ and $2009 \equiv -1 \pmod{3}$. So, equation (5) can be rewritten in the form

$$2009^{2k+1} = (3^n - x)(3^n + x) \quad (7)$$

Since $3 \nmid x$ (because $3 \nmid 2009$), we get

$$\{3^n - x, 3^n + x\} = \pm\{1, 2009^{2k+1}\}$$

Hence, we may suppose that $3^n - x = 1$ and $3^n + x = 2009^{2k+1}$. Thus

$$2 \cdot 3^n - 2009^{2k+1} = 1 \quad (8)$$

We point out that the above equation has no solution when $n = 2k + 1$. This fact is an immediate consequence of a result due to Bennett [9]: for any positive integer $a$, the equation

$$(a + 1)x^n - ay^n = 1,$$

in integers $x \geq 1$, $y \geq 1$, $n \geq 3$, has no solution other than given by $x = y = 1$.

For the remaining cases ($n \neq 2k + 1$), we shall use bounds for linear forms in three logarithms of algebraic numbers (for more details on transcendental methods to Diophantine equations we refer the reader to [30]).

First, on dividing Eq. (8) through by $2009^{2k+1}$, we get

$$2 \cdot 3^n \cdot 2009^{-(2k+1)} - 1 = 2009^{-(2k+1)}$$

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Let \( \Lambda = (2k + 1) \log(1/2009) - n \log(1/3) + \log 2 \), then the previous equality becomes \( e^\Lambda - 1 = 2009^{-(2k+1)} > 0 \) and so \( \Lambda > 0 \). Therefore \( \Lambda < e^\Lambda - 1 = 2009^{-(2k+1)} \) which yields

\[
\log \Lambda < -(2k + 1) \log 2009 \tag{9}
\]

Now, we will apply Lemma 1. Take

\[
\alpha_1 = 1/2009, \quad \alpha_2 = 1/3, \quad \alpha_3 = 2, \quad b_1 = 2k + 1, \quad b_2 = -n, \quad b_3 = 1.
\]

Observe that \( \mathbb{Q}(\alpha_1, \alpha_2, \alpha_3) = \mathbb{Q} \) and then \( D = 1 \). Surely, we can take \( A_1 = \log 2009, \ A_2 = \log 3 \) and \( A_3 = \log 2 \).

Note that

\[
\max\{1, \max\{|b_j|A_j/A_1; 1 \leq j \leq 3\}\} = \max\{2k + 1, n \log 3/\log 2009\},
\]

and then it suffices to choose \( B = 2k + 1 \) as

\[
2 \cdot 3^n = 2009^{2k+1} + 1 < 2 \cdot 2009^{2k+1} \text{ and then } n \log 3 < (2k + 1) \log 2009.
\]

Since, for \( D = 1 \), it holds that \( C_1 < 9.7 \cdot 10^9 \), Lemma 1 yields

\[
\log |\Lambda| > -56.2 \cdot 10^9 \log(4.08(2k + 1)). \tag{10}
\]

Combining the estimates (9) and (10), we get

\[
56.2 \cdot 10^9 \log(4.08(2k + 1)) > (2k + 1) \log 2009,
\]

and this inequality implies \( k < 2 \cdot 10^{11} \) (for the sake of preciseness \( k < 101389315227 \)).

**Reducing the bound**

Since \( 0 < \Lambda < 2009^{-2k-1} \), we have that

\[
0 < (2k + 1) \log \alpha_1 - n \log \alpha_2 + \log \alpha_3 < 2009^{-2k}.
\]

On dividing through by \( \log \alpha_2 \), we get

\[
0 < (2k + 1) \gamma - n + \mu < 2009^{-2k}, \tag{11}
\]

with \( \gamma = \log \alpha_1/\log \alpha_2 \) and \( \mu = \log \alpha_3/\log \alpha_2 \).

Surely \( \gamma \) is an irrational number\(^a\) (because 2009 and 3 are multiplicatively independent). So, let us denote \( p_\ell/q_\ell \) be the \( \ell \)th convergent of its continued fraction.

In order to reduce our bound on \( k \) (which is too large!), we will use the Lemma 2.

For that, take \( M = 2 \cdot 10^{11} \). Since

\[
p_{27} = 2478237449400, \quad q_{27} = 3579857528251,
\]

\(^{a}\)Actually, this number is transcendental by Gelfond-Schneider theorem: if \( \alpha \) and \( \beta \) are algebraic numbers, with \( \alpha \neq 0 \) or 1, and \( \beta \) irrational, then \( \alpha^\beta \) is transcendental.
then \( q_{27} \geq 3579857528251 > 1.2 \cdot 10^{12} = 6M \). Moreover, a straight calculation gives

\[ M \parallel q_{27} \gamma \parallel = 0.02760... < 0.02, \]

and

\[ \parallel q_{27} \mu \parallel = 0.33016... > 0.34 \]

Hence

\[ \epsilon = \parallel \mu q_{27} \parallel - M \parallel \gamma q_{27} \parallel > 0.34 - 0.02 = 0.32 \]

Thus all the hypotheses of the Lemma 2 are satisfied with \( A = 1 \) and \( B = 2009^2 \). It follows from that lemma that there is no solution of the Diophantine equation (7) in the range

\[ \left\lceil \frac{\log(Aq_{27}/\epsilon)}{\log B} \right\rceil + 1, M \right) = \left[ 115, 2 \cdot 10^{11} \right) \]

For the remaining possibilities (that is \( k < 115 \)), we define a function \( T : \mathbb{N} \rightarrow \mathbb{R} \) given by

\[ T(s) := \frac{\log \left( \frac{2009^{2s+1}+1}{2} \right)}{\log 3} \]

Thus in view of the relation in (8), if the equation (7) has solution for a certain \( k \), then \( T(k) \) must be an integer number. To finish, we use Mathematica to print all the values of this function, for \( 1 \leq k \leq 114 \). This task took less than one second on a 1.86 GHz Pentium Core Duo. Finally, we convince ourselves that \( T(k) \) is never an integer number in the obtained range. This completes the proof.

\[ \square \]

### 3 The proof of Theorem 3

Note that if \( m = n \), then \( 3^{2n} - 2 \cdot 3^n + 1 = (3^n - 1)^2 \). If \( k \) is positive, then \( (n, m, k) = (n, n, 3^n - 1) \) is solution for (3) for all positive integer \( n \). Our goal is to prove that this one is the only possibility.

For that, in order to facilitate our work, we shall denote \( \delta_{m,n} = 3^{2n} - 2 \cdot 3^n + 1 \) and let \( m, n, k \) be positive integer numbers such that \( \delta_{m,n} = k^2 \). First, take \( p = 3^n + k \) and \( q = 3^n - k \). So, we have \( p > q \geq 1 \), \( p + q = 2 \cdot 3^n \) and \( pq = 2 \cdot 3^n - 1 \). Now, if \( x = 3^m - 1 \) and \( y = 3^n - q = p - 3^n = k \), we get

\[ x = 3^m - 1 = pq - 3^m \] and \( y = 3^n - q = p - 3^n = k \)

yielding

\[ (3^m + x)(3^m - x) = pq = (3^n + y)(3^n - y) \]

Thus \( (x, n) \) and \( (y, m) \) are solutions of the equation (4) with \( C = pq \). Hence we apply the Theorem 1 to get \( m = n \) and this completes our proof. \[ \square \]
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