Yet Another “Direct” Proof of the Uncountability
of the Transcendental Numbers

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The most known proof of uncountability of the transcendental numbers is based on proving that \( \mathbb{A} \) is countable and concluding that \( \mathbb{R} \setminus \mathbb{A} \) is uncountable since \( \mathbb{R} \) is. Very recently, J. Gaspar [1] gave a nice “direct” proof that the set of transcendental numbers is uncountable. In this context, the word direct means a proof which does not follow the previous steps. However, we point out that his proof is based on the transcendence of \( \pi \) which is, to the best of the author’s knowledge, proved by an indirect argument. In this note, in the spirit of Gaspar, we present a “direct” proof of the following stronger result.

**Theorem.** There exist uncountable many algebraically independent real numbers. So the set of the transcendental real numbers is uncountable.

**Proof.** Let \( \mathcal{B} \) be a transcendence basis (which exists by Zorn’s lemma) of the field extension \( \mathbb{R}/\mathbb{Q} \). If \( \mathcal{B} \) were countable, then \( \mathbb{Q}(\mathcal{B}) \) would be countable (because its elements are of the form \( P(\vec{b})/Q(\vec{c}) \) with \( n \in \mathbb{N} , P, Q \in \mathbb{Q}[X_1, \ldots , X_n] \) and \( \vec{b}, \vec{c} \in \mathcal{B}^n \)), so \( \mathbb{R} \) would be countable (because its elements are roots of polynomials in \( \mathbb{Q}(\mathcal{B})[X] \setminus \{0\} \) and there would be only countable roots), which is false. The elements of \( \mathcal{B} \) are uncountable many algebraically independent real numbers. ■

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**References**


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