On the Diophantine equation \( x^2 + 2^\alpha 5^\beta 17^\gamma = y^n \)

Hemar Godinho, Diego Marques and Alain Togbé

Abstract. In this paper, we find all solutions of the Diophantine equation \( x^2 + 2^\alpha 5^\beta 17^\gamma = y^n \) in positive integers \( x, y \geq 1, \alpha, \beta, \gamma, n \geq 3 \) with \( \gcd(x, y) = 1 \).

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Keywords. Diophantine equation, exponential equation, primitive divisor theorem.

1. Introduction

The interest for the diophantine equation

\[ x^2 + C = y^n, \quad x \geq 1, \quad y \geq 1, \quad n \geq 3 \quad (1.1) \]

started with a paper due to Lebesgue [18] and dating back to 1850, where he proved that the above equation has no solutions for \( C = 1 \). More recently, other values of \( C \) were considered. Tengely [24] solved the equation with \( C = b^2 \) and \( 3 \leq b \leq 501 \). The case where \( C = p^k \), a power of a prime number, was studied in [7, 16, 17] for \( p = 2 \), in [5, 6, 19] for \( p = 3 \), in [1, 2] for \( p = 5 \), and in [22] for \( p = 7 \). For arbitrary primes, some advances can be found in [4]. In [9], the cases with \( 1 \leq C \leq 100 \) were completely solved. The solutions for the cases \( C = 2^a \cdot 3^b \), \( C = 2^a \cdot 5^b \), and \( C = 5^a \cdot 13^b \), when \( x \) and \( y \) are coprime, can be found in [3, 20, 21], respectively. Recent progress on the subject were made in the cases \( C = 5^a \cdot 11^b \), \( C = 2^a \cdot 11^b \), \( C = 2^a \cdot 3^b \cdot 11^c \), \( C = 2^a \cdot 5^b \cdot 13^c \) and can be found in [10, 11, 12, 14].

In this paper, we are interested in solving the Diophantine equation

\[ x^2 + 2^\alpha 5^\beta 17^\gamma = y^n, \quad \gcd(x, y) = 1, \quad x, y \geq 1, \quad \alpha, \beta, \gamma \geq 0, \quad n \geq 3 \quad (1.2) \]

Our result is the following.

Theorem 1.1. The equation (1.2) has no solution except for:

\[ \begin{align*}
    n = 3 & \quad \text{the solutions given in Table 1;} \\
    n = 4 & \quad \text{the solutions given in Table 2;} \\
    n = 5 & \quad (x, y, \alpha, \beta, \gamma) = (401, 11, 1, 3, 0)
\end{align*} \]
\( n = 6 \) \( (x, y, \alpha, \beta, \gamma) = (7, 3, 3, 1, 1), (23, 3, 3, 2, 0) \);
\( n = 8 \) \( (x, y, \alpha, \beta, \gamma) = (47, 3, 8, 0, 1), (79, 3, 6, 1, 0) \).

One can deduce from the above result the following corollary.

**Corollary 1.2.** *The equation*
\[ x^2 + 5^k 17^l = y^n, \quad x \geq 1, \quad y \geq 1, \quad \gcd(x, y) = 1, \quad n \geq 3, \quad k \geq 0, \quad l \geq 0 \quad (1.3) \]
*has only the solution*
\[ (x, y, k, l, n) = (94, 21, 2, 1, 3), \quad (2034, 161, 3, 2, 3), \quad (8, 3, 0, 1, 4). \]

Therefore, our work extends that of Pink and Rábai [23].

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**Table 1.** Solutions for \( n = 3 \).
On the Diophantine equation \( x^2 + 2^\alpha 5^\beta 17^\gamma = y^n \)

### Table 2. Solutions for \( n = 4 \).

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2. **The case \( n = 3 \)**

**Lemma 2.1.** When \( n = 3 \), all the solutions to equation (1.2) are given in Table 1.

For \( n = 6 \), we have \((x, y, \alpha, \beta, \gamma) = (7, 3, 3, 1, 1), (23, 3, 3, 2, 0)\).

**Proof.** Equation (1.2) can be rewritten as

\[
\left( \frac{x}{z^3} \right)^2 + A = \left( \frac{y}{z^2} \right)^3, \tag{2.1}
\]

where \( A \) is sixth-power free and defined implicitly by \( 2^\alpha 5^\beta 17^\gamma = Az^6 \). One can see that \( A = 2^{\alpha_1} 5^{\beta_1} 17^{\gamma_1} \) with \( \alpha_1, \beta_1, \gamma_1, \in \{0, 1, 2, 3, 4, 5\} \). We thus get

\[
V^2 = U^3 - 2^{\alpha_1} 5^{\beta_1} 17^{\gamma_1}, \tag{2.2}
\]

with \( U = y/z^2 \), \( V = x/z^3 \) and \( \alpha_1, \beta_1, \gamma_1, \in \{0, 1, 2, 3, 4, 5\} \). We need to determine all the \( \{2, 5, 17\} \)-integral points on the above 216 elliptic curves. Recall that if \( S \) is a finite set of prime numbers, then an \( S \)-integer is rational number \( a/b \) with coprime integers \( a \) and \( b \), where the prime factors of \( b \) are in \( S \). We use MAGMA [13] to determine all the \( \{2, 5, 17\} \)-integer points on the above elliptic curves. Here are a few remarks about the computations:

1. We eliminate the solutions with \( UV = 0 \) because they yield to \( xy = 0 \).
2. We consider only solutions such that the numerators of \( U \) and \( V \) are coprime.
3. If \( U \) and \( V \) are integers then \( z = 1 \). So \( \alpha_1 = \alpha, \beta_1 = \beta, \) and \( \gamma_1 = \gamma \).
4. If \( U \) and \( V \) are rational numbers which are not integers, then \( z \) is determined by the denominators of \( U \) and \( V \). The numerators of these rational numbers give \( x \) and \( y \). Then \( \alpha, \beta, \gamma \) are computed knowing that \( 2^\alpha 5^\beta 17^\gamma = Az^6 \).
Therefore, we first determine \((U, V, \alpha_1, \beta_1, \gamma_1)\) and then we use the relations
\[
U = \frac{y}{z^2}, \quad V = \frac{x}{z^3}, \quad 2^{\alpha_5}5^\beta 17^\gamma = Az^6,
\]
to find the solutions \((x, y, \alpha, \beta, \gamma)\) listed in Table 1.

For \(n = 6\), equation
\[
x^2 + 2^{\alpha_5}5^\beta 17^\gamma = y^6
\]
becomes equation
\[
x^2 + 2^{\alpha_5}5^\beta 17^\gamma = (y^2)^3.
\]

We look in the list of solutions of equation Table 1 and observe that the only solutions in Table 1 whose \(y\) is a perfect square. Therefore, the only solutions to equation (1.2) for \(n = 6\) are the two solutions listed in Theorem 1.1. This completes the proof of Lemma 2.1.

\section{The case \(n = 4\)}

Here, we have the following result.

\begin{lemma}
If \(n = 4\), then the only solutions to equation (1.2) are given in Table 2.

If \(n = 8\), then the only solution to equation (1.2) is \((x, y, \alpha, \beta, \gamma) = (47, 3, 8, 0, 1), (79, 3, 6, 1, 0)\).
\end{lemma}

\begin{proof}
Equation (1.2) can be written as
\[
\left(\frac{x}{z^2}\right)^2 + A = \left(\frac{y}{z}\right)^4,
\]
where \(A\) is fourth-power free and defined implicitly by \(2^{\alpha_1}5^\beta 17^\gamma = Az^4\). One can see that \(A = 2^{\alpha_1}5^\beta 17^{\gamma_1}\) with \(\alpha_1, \beta_1, \gamma_1 \in \{0, 1, 2, 3\}\). Hence, the problem consists of determining the \(\{2, 5, 17\}\)-integer points on the totality of the 64 elliptic curves
\[
V^2 = U^4 - 2^{\alpha_1}5^\beta 17^{\gamma_1},
\]
with \(U = y/z, \ V = x/z^2\) and \(\alpha_1, \beta_1, \gamma_1 \in \{0, 1, 2, 3\}\). Here, we use again \textsc{Magma} [13] to determine the \(\{2, 5, 17\}\)-integer points on the above elliptic curves. As in Section 2, we first find \((U, V, \alpha_1, \beta_1, \gamma_1)\), and then using the coprimality conditions on \(x\) and \(y\) and the definition of \(U\) and \(V\), we determine all the corresponding solutions \((x, y, \alpha, \beta, \gamma)\) listed in Table 2.

Looking in the list of solutions of equation Table 2, we observe the 8 solutions in Table 2 whose values for \(y\) are perfect squares. Thus, the only solutions to equation (1.2) with \(n = 8\) are those listed in Theorem 1.1. This concludes the proof of Lemma 3.1.
\end{proof}
4. The case \( n \geq 5 \)

The aim of this section is to determine all solutions of equation (1.2), for \( n \geq 5 \) and to prove its unsolubility for \( n = 7 \) and \( n \geq 9 \). The cases when \( n \) is of the form \( 2^a 3^b \) were treated in previous sections. So, apart from these cases, in order to prove that (1.2) has no solution for \( n \geq 7 \), it suffices to consider \( n \) prime. In fact, if \((x, y, \alpha, \beta, \gamma, n)\) is a solution for (1.2) and \( n = pk \), where \( p \geq 7 \) is prime and \( k > 1 \), then \((x, y^k, \alpha, \beta, \gamma, p)\) is also a solution. So, from now on, \( n \) will denote a prime number.

**Lemma 4.1.** The Diophantine equation (1.2) has no solution with \( n \geq 5 \) prime except for

\[
  n = 5 \quad (x, y, \alpha, \beta, \gamma) = (401, 11, 1, 3, 0);
\]

Proof. Let \((x, y, \alpha, \beta, \gamma, n)\) be a solution for (1.2). We claim that \( y \) is odd. In fact, if \( y \) is even and since \( \gcd(x, y) = 1 \), one has that \( x \) is odd, and then \(-2^a 5^b 17^\gamma \equiv x^2 - y^n \equiv 1 \pmod{4} \), but this contradicts the fact that \(-2^a 5^b 17^\gamma \equiv 0, 2 \text{ or } 3 \pmod{4} \). Now, write equation (1.2) as \( x^2 + dz^2 = y^n \), where

\[
  d = 2^{\alpha - 2[\alpha/2]} 5^{\beta - 2[\beta/2]} 17^{\gamma - 2[\gamma/2]},
\]

and \( z = 2^{[\alpha/2]} 5^{[\beta/2]} 17^{[\gamma/2]} \). Since \( x - 2[x/2] \in \{0, 1\} \), we have

\[
  d \in \{1, 2, 5, 10, 17, 34, 85, 170\}.
\]

We then factor the previous equation over \( \mathbb{K} = \mathbb{Q}[i\sqrt{d}] = \mathbb{Q}[-\sqrt{d}] \) as

\[
  (x + i\sqrt{d}z)(x - i\sqrt{d}z) = y^n.
\]

Now, we claim that the ideals \((x + i\sqrt{d}z)\mathcal{O}_\mathbb{K}\) and \((x - i\sqrt{d}z)\mathcal{O}_\mathbb{K}\) are coprime. If this is not the case, there must exist a prime ideal \( \mathfrak{p} \) containing these ideals. Therefore, \( x \pm i\sqrt{d}z \) and \( y^n \) (and so \( y \)) belong to \( \mathfrak{p} \). Thus \( 2x \in \mathfrak{p} \) and hence either \( 2 \) or \( x \) belongs to \( \mathfrak{p} \). Since \( \gcd(2, y) = \gcd(x, y) = 1 \), then \( 1 \) belongs to the ideals \((2, y)\) and \((x, y)\), then \( 1 \in \mathfrak{p} \) leading to an absurdity of \( \mathfrak{p} = \mathcal{O}_\mathbb{K} \). By the unique factorization of ideals, it follows that \((x + i\sqrt{d}z)\mathcal{O}_\mathbb{K} = j^n\), for some ideal \( j \) of \( \mathcal{O}_\mathbb{K} \). Using the Mathematica command

\[
  \text{NumberFieldClassNumber}[\text{Sqrt}[-d]]
\]

we obtain that the class number of \( \mathbb{K} \) is either \( 1, 2, 4 \) or \( 12 \) and so coprime to \( n \), then \( j \) is a principal ideal yielding

\[
  x + i\sqrt{d}z = \varepsilon \eta^n, \quad (4.1)
\]

for some \( \eta \in \mathcal{O}_\mathbb{K} \) and \( \varepsilon \) a unit of \( \mathbb{K} \). Since the group of units of \( \mathbb{K} \) is a subset of \( \{-1, 1\} \) and \( n \) is odd, then \( \varepsilon \) is a \( n \)-th power. Thus, (4.1) can be reduced to \( x + i\sqrt{d}z = \eta^n \). Since \( \mathbb{K} \) is an imaginary quadratic field and \(-d \not\equiv 1 \pmod{4} \), then \( \{1, i\sqrt{d}\} \) is an integral basis and we can write \( \eta = u + i\sqrt{d}v \), for some integers \( u \) and \( v \). We then get

\[
  \frac{\eta^n - \overline{\eta}^n}{\eta - \overline{\eta}} = \frac{2^{[\alpha/2]} 5^{[\beta/2]} 17^{[\gamma/2]}}{v}, \quad (4.2)
\]
where, as usual, $\overline{w}$ denotes the complex conjugate of $w$.

Let $(L_m)_{m \geq 0}$ be the Lucas sequence given by

$$L_m = \frac{\eta^m - \overline{\eta}^m}{\eta - \overline{\eta}}$$

for $m \geq 0$.

We recall that the Primitive Divisor Theorem for Lucas sequences ensures for primes $n \geq 5$, that there exists a primitive divisor for $L_n$, except for the finitely many (defective) pairs $(\eta, \overline{\eta})$ given in Table 1 of [8] (a primitive divisor of $L_n$ is a prime that divides $L_n$ but does not divide $(\eta - \overline{\eta})^2 \prod_{j=1}^{n-1} L_j$).

And a helpful property of a primitive divisor $p$ is that $p \equiv \pm 1 \pmod{n}$.

For $n = 5$, we find in Table 1 in [8] that $L_5$ has a primitive divisor except for $(u,d,v) = (1,10,1)$ which leads to a number $\eta = 1 + i\sqrt{10} \in \mathbb{Q}[i\sqrt{10}]$ ($d = 10$ is one of the possible values of $d$ described in the beginning of this proof), which gives the solution with $n = 5$.

Apart from this case, let $p$ be a primitive divisor of $L_n$, $n \geq 7$. The identity (4.2) implies that $p \in \{2, 5, 17\}$ and so $p = 17$, since $p \not\equiv \pm 1 \pmod{n}$, for $p = 2, 5$. Hence, $n$ is a prime dividing $17 \pm 1$ and so $n = 2$ or 3 which contradicts the fact that $n \geq 7$. This completes the proof of Theorem 1.1.

\[\square\]

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On the Diophantine equation $x^2 + 2^a 5^b 17^c = y^n$


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